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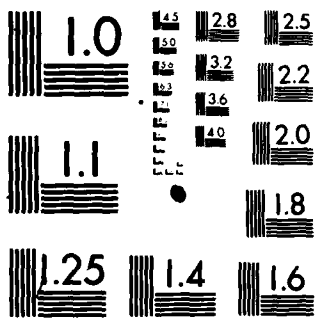

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Central Limit Theorems in the Area of Large Deviations for Some Dependent Random Variables

by

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Abstract

A triangular array of dependent random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ whose joint distribution is given by $dQ_n(\underline{x}) = z_n^{-1}\exp[-H_n(\underline{X})]\Pi dP(x_j)$, where $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, z_n is the normalizing constant and P is a probability measure on $\mathbb{R}$ has been used to describe the distribution of magnetic spins in a body. Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$ be the tctal magnetism present in the body. For certain forms of the function H, Ellis and Newman (Z. Wahrscheinlichkeistheorie und Verw. Gebiete 44 (1978) 117-139) and Jong-Woo Jeon and Sethuraman (IMS Bulletin (1978) Abstract #165-116) showed that under appropriate conditions on P, there exists an integer $r \geq 1$ such that $S_n/n^{1-\frac{1}{2r}}$ converges in distribution to a random variable which is Gaussian for $r = 1$ and non-Gaussian for $r \geq 2$. In this paper utilizing the large deviation local limit theorems for arbitrary sequences of random variables of Chaganty and Sethuraman (Dept. of Stat., FSU, Tech. Report M630) we obtain similar central limit theorems for a wider class of functions H_n, thus generalizing the results of the previous authors.

1. Introduction

In this paper we obtain central limit theorems for some dependent random variables which are used to describe the distribution of magnetic spins present in a ferromagnet crystal. A ferromagnet crystal consists of a large number of sites. At site i there is some amount of magnetic spin present, ~~which~~ will be denoted by $X_i^{(n)}$, $i = 1, \ldots, n$, where n is a positive integer. The magnetic spin present at any site interacts with the magnetic spins at its neighboring sites and hence gives rise to some dependency among the magnetic spin present. the $X_i^{(n)}$'s. In the Ising model, the joint distribution, at a fixed temperature $T > 0$, of the spin random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ is given by

$$dQ_n(\underline{x}) = z_n^{-1} \exp\left[-\frac{H_n(\underline{x})}{T}\right] \prod_{j=1}^{n} dP(x_j) \tag{1.1}$$

where $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and P is a probability measure on $\mathbb{R}$ with mean 0 and variance 1. The function $H_n(\underline{x})$ is known as the Hamiltonian and it represents the energy of the crystal at the configuration $\underline{x}$, z_n is the normalizing constant which is also known as the partition function. In many cases explicit evaluation of z_n is very difficult and physicists usually try to evaluate the limiting free spin per state $\psi(T)$, at the temperature T, as defined below:

$$\psi(T) = -\lim_{n\to\infty} \frac{1}{n} \log z_n. \tag{1.2}$$

For some particular types of Hamitonians it was shown by physicists that there exists a temperature level T_c such that the function $\psi(T)$ is infinite or finite according as $T < T_c$ or $T \geq T_c$ (see Kac (1968)). T_c is the critical temperature at which a phase transition occurs. As pointed out by Ellis and Newman (1978) the existence of the critical temperature can be demonstrated in yet another way. For $T > T_c$, the model shows that there is a weak dependence among the random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ and a standard central limit theorem is valid for $S_n/\sqrt{n}$. However for $T = T_c$, there exists a $\delta \in (1, 2)$ such that $S_n/n^{\delta/2}$ converges to a non-Gaussian limit and for $T < T_c$ due to the strong dependence of the $X_i^{(n)}$'s, the random variables tend to cluster in several ergodic components. One can show that the central limit theorem is valid on each of the components. This is the approach that we take in this paper. In Section 2, we consider a special case for the Hamiltonian by setting it to be equal to $-\frac{1}{2n}\Sigma\Sigma\, x_i x_j$. This is known as the Curie-Weiss model. The asymptotic distribution of S_n for this model when P is symmetric Bernoulli is obtained by Simon and Griffiths (1973). In a two paper series, Ellis and Newman (1977, 1978) extended Theorem 2.1 of Simon and Griffiths to the class of probability measures L_0, defined in 2.2, and subsequently to the class L, defined in 2.4. These extensions are stated in Theorems 2.3 and 2.8. Recently Jong-Woo Jeon (1979) in his Ph.D. dissertation gave a simpler and statistically motivated proof of Theorem 2.8 and used the technique to obtain similar limit theorems for a wider class of Hamiltonians. His results as well as the results of the previous authors are stated precisely in Section 2. The goal of this paper is to extend Theorem 2.11 of Jong-Woo Jeon (1979) further for a larger class of Hamiltonians. Our main result, Theorem 3.4, is stated in Section 3. The proof of Theorem 3.4 rests on recent large deviation local limit theorems of Chaganty and Sethuraman (1982). We also state these results, Theorems 3.1 and 3.2, in Section 3.

We now briefly give our reasons for calling theorems on the asymptotic distribution of S_n under Q_n, defined in (1.1), as limit theorems in the area of large deviations. A standard technique in statistics literature is to first obtain the asymptotic distribution of S_n under P_n, where

$$dP_n(\underline{x}) = \prod_{j=1}^{n} dP(x_j) \tag{1.3}$$

and then use contiguity arguments, as in LeCam (1960) to obtain the asymptotic distribution under Q_n. This technique breaks down completely in this case. For the various models considered in Physics which are described in greater detail in Sections 2 and 3,

$$\left|L_n(\underline{x})\right| = \left|\log \frac{dQ_n(\underline{x})}{dP_n(\underline{x})}\right| = \left|\frac{H_n(\underline{x})}{T} + \log z_n\right|$$

converges to ∞ in probability under P_n and thus contiguity arguments are not applicable here. Under P_n, $S_n/\sqrt{n}$ has a limiting normal distribution. Also, under P_n, $|L_n(\underline{x})|$ is small in the area of ordinary deviations of S_n, that is, when $S_n/\sqrt{n}$ is finite, while it is large otherwise. Thus from the point of view of P_n, we are looking for the asymptotic distribution of S_n, when P_n is modified by $L_n(\underline{x})$, which is substantially different from 1 in the area of large deviations of S_n. This view point helps in a statistically motivated proof of the asymptotic distribution of S_n under Q_n and describes the background behind the title of this paper. One should also note that the normalizing factor on S_n in its asymptotic distribution under Q_n is different from the corresponding factor under P_n.

2. A Brief Summary of Curie-Weiss Model and Its Extensions.

In a ferromagnetic system with only isotropic pair interactions and with no external magnetic field, the form of the Hamiltonian, H_n, may be taken as $H_n(x_1, \ldots, x_n) = -\frac{1}{2}\Sigma\Sigma\, a_{ij}x_ix_j$, where $a_{ij} \geq 0$. If it is assumed further that $a_{ij} = \frac{1}{n}$ for all i and j, that is to say that each spin interacts equally with every other spin with strength $\frac{1}{n}$ and P is taken to be symmetric Bernoulli, i.e., $P(-1) = P(1) = \frac{1}{2}$, one obtains the Curie-Weiss model. Replacing P by $P_T(x) = P(x\sqrt{T})$, we get

$$dQ_n(\underset{\sim}{x}) = z_n^{-1} \exp[s_n^2/2n]\, \Pi dP(x_j), \text{ where } s_n = x_1 + \ldots + x_n. \quad (2.1)$$

This model has the advantage, that the limiting free spin per site can be solved exactly. The existence of the critical temperature and phase transition for this model was demonstrated by Kac(1968) and the asymptotic distribution for the total magnetism, S_n, for this model was obtained by Simon and Griffiths (1973) which is contained in Theorem 2.1.

Theorem 2.1. (Simon and Griffiths). Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables whose joint distribution is given by (2.1) and P be symmetric Bernoulli. Then $S_n/n^{3/4}$ converges in distribution to a random variable whose density function is proportional to $\exp(-y^4/12)$.

Theorem 2.1 was extended to the class of probability measures L_0, which is defined below, by Ellis and Newman (1977).

Definition 2.2. Let L_0 be the class of symmetric probability measures P on $\mathbb{R}$ such that

$$\int \exp(x^2/2)dP(x) < \infty \tag{2.2}$$

and

$$\phi(t) = \int e^{tx}dP(x) < e^{t^2/2} \quad \text{for } t \neq 0. \tag{2.3}$$

Let $\psi(t) = \log\phi(t)$ be the cumulant generating function of P. Since P is symmetric, the Taylor series expansion of $\psi(t)$ about the origin consists of even powers of t,

$$\psi(t) = \sum_{s=1}^{\infty} c_{2s}t^{2s}/(2s)!, \tag{2.4}$$

where c_{2s} is the $2s$th cumulant of P and the series converges in a neighborhood of the origin. Let the index r be defined as

$$r = \begin{cases} 1 \quad \text{if } c_2 = 1 \\ \min\{s > 1: \ c_{2s} \neq 0\} \text{ if } c_2 = 1. \end{cases} \tag{2.5}$$

It is easily verified that the symmetric Bernoulli belongs to the class L_0 with the corresponding value of r equal to 2. Thus the following theorem due to Ellis and Newman (1977) extends Theorem 2.1 to a larger class of probability measures. Let Y_r, $r \geq 1$, be a sequence of random variables with density function $p_r(y)$, where

$$P_r(y) = \begin{cases} d_r \exp[-c_{2r}\, y^{2r}/(2r)!] & \text{if } r \geq 2 \\ N(0,\ (1 - c_2)/c_2) & \text{if } r = 1, \end{cases} \tag{2.6}$$

and d_r is the normalizing constant.

Theorem 2.3. (Ellis and Newman). Let $P \in L_0$ and the index r be defined by (2.5). Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by (2.1). Then

$$\frac{S_n}{n^{1-1/2r}} \xrightarrow{d} Y_r . \tag{2.7}$$

Ellis and Newman (1978) further extended Theorem 2.3 to a bigger class of probability measures L than L_0 by removing the assumption of symmetry and Condition (2.3). The class L is defined below.

Definition 2.4. Let L be the class of probability measures P on $\mathbb{R}$ such that

$$\int \exp(x^2/2)\,dP(x) < \infty. \tag{2.8}$$

Fix $P \in L$. Let $\psi(t)$ be the c.g.f. of P and set the function $G(t) = t^2/2 - \psi(t)$, for $t \in \mathbb{R}$.

Definition 2.5. A real number m is said to be a global minimum for G if $G(t) \geq G(m)$ for all t.

Definition 2.6. A global minimum m for G is said to be of type r if

$$G(t+m) - G(m) = c_{2r} t^{2r}/(2r)! + o(|t|^{2r}) \text{ as } t \to 0, \tag{2.9}$$

where $c_{2r} = G^{(2r)}(m)$ is strictly positive.

Definition 2.7. A probability measure P is said to be pure if G has a unique global minimum.

With these definitions we are now in a position to state further generalization of Theorem 2.3 also due to Ellis and Newman (1978).

Theorem 2.8. (Ellis and Newman). Let $P \in L$ be pure and m be the unique global minimum of type r. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by (2.1). Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Then

$$\frac{S_n - nm}{n^{1-\frac{1}{2r}}} \xrightarrow{d} Y_r, \qquad (2.10)$$

where Y_r is defined by (2.6).

An alternate proof of the above theorem was given by Jong-Woo Jeon (1979). Using the technique of this new proof he was able to obtain similar limit theorems for a wider class of Hamiltonians. We present his results after making a few observations. Note that the moment generating function m(t) of the standard normal is given by $m(t) = e^{t^2/2}$. Then we can write (2.1) as

$$dQ_n(\underset{\sim}{x}) = z_n^{-1}[m(s_n/n)]^n \, \Pi dP(x_j), \quad \text{where } s_n = x_1 + \ldots + x_n. \qquad (2.11)$$

One might ask the question whether it is possible to obtain limit theorems of the type (2.10) when m is replaced by the moment generating function ϕ_u of a random variable U, not necessarily standard normal. This is precisely the question that was raised and answered in the affirmative by Jong-Woo Jeon (1979).

Fix a random variable U with E(U) = 0 and whose m.g.f. ϕ_u is finite in a neighborhood of the origin. Assume that the density function of U is bounded. Let $\gamma_u(s) = \sup_t[st - \log \phi_u(t)]$ be the large deviation rate of U. For a probability measure P, let

$$G_u(s) = \gamma_u(s) - \psi(s), \quad \text{where } \psi(s) = \log \int e^{sx} dP(x). \tag{2.12}$$

Definition 2.9. Let L_u be the class of probability measures P such that

$$\int e^{-G_u(s)} ds < \infty \tag{2.13}$$

and

$$\int \phi_u(s)\, dP(s) < \infty. \tag{2.14}$$

Definition 2.10. A global minimum m for G_u is said to be of Type r if

$$G_u(s) - G_u(m) = c_{2r} s^{2r}/(2r)! + o(|s|^{2r}), \quad \text{as } |s| \to 0, \tag{2.15}$$

where $c_{2r} = G_u^{(2r)}(m) > 0$.

The following Theorem 2.11 generalizes Theorem 2.8 to a larger class of Hamiltonians.

Theorem 2.11. (Jong-Woo Jeon). Let $P \in L_u$ and $X_j^{(n)}$, j = 1, ..., n, be a triangular array of random variables with joint distribution given by

$$dQ_n(\underset{\sim}{x}) = z_n^{-1}\, \phi_u^n(s_n/n)\, \Pi dP(x_j), \tag{2.16}$$

where $\underline{x} = (x_1, \ldots, x_n)$, $s_n = x_1 + \ldots + x_n$ and z_n is the normalizing constant. Assume that G_u has the unique global minimum of type r at the origin. Then

$$\frac{S_n}{n^{1-\frac{1}{2r}}} \xrightarrow{d} Y_r, \tag{2.17}$$

where Y_r is defined by (2.6).

3. Further Extensions of the Curie-Weiss Model.

In this section we propose to extend further Theorem 2.11 by enlarging the class of Hamiltonians as well as the class of probability measures L_u. The large deviation local limit theorems for arbitrary sequence T_n, $n \geq 1$, of random variables of Chaganty and Sethuraman (1982) are the key tools which make this extension of Theorem 2.11 possible. The Hamiltonian, H_n, in our generalized model (3.7) is taken to be the cumulant generating function of these random variables T_n. We state the main Theorem 3.4 after presenting the large deviation local limit theorems 3.1 and 3.2.

Let $\{T_n, n \geq 1\}$ be a sequence of non-lattice random variables with c.f. $\phi_n(z)$ which is analytic and non-vanishing for z in $\Omega = \{z: |\mathrm{Real}(z)| < a\}$ with $a > 0$. Let $I = (-a,a)$ and $I_1 = (-a_1, a_1)$, where $0 < a_1 < a$. Let

$$\psi_n(z) = \frac{1}{n} \log \phi_n(z) \quad \text{for } z \in \mathbb{C} \text{ and} \tag{3.1}$$

$$\gamma_n(u) = \sup_{s \in I}[su - \psi_n(u)]. \tag{3.2}$$

Assume that $E(T_n)/n = m$ and $Var(T_n)/n = \sigma^2$, $\forall n \geq 1$. Let $\{m_n\}$ be a sequence of real numbers converging to m as $n \to \infty$ such that $n^\delta|m_n - m| > 1$, for $0 < \delta < 1$. Let $G_{n,\tau}(t) = \psi_n(\tau)+it\ m_n-\psi_n(\tau+it)$, for $\tau \in I_1$. The following theorem, which provides an asymptotic expansion for the density function k_n of T_n/n in terms of the large deviation rate γ_n, is due to Chaganty and Sethuraman (1982).

Theorem 3.1. Assume the following conditions for T_n:

(A). There exists $\beta > 0$ such that $|\psi_n(z)| < \beta\ \forall\ z \in \Omega$, $\forall n \geq 1$.

(B). There exists $\alpha > 0$ and $\tau_n \in I_1$ such that $\psi_n'(\tau_n) = m_n$ and $\psi_n''(\tau_n) \geq \alpha$, $\forall\ \tau \in I_1$, $\forall\ n \geq 1$.

(C). There exists $\eta > 0$ such that for any $0 < \delta < \eta$,

$$\inf_{|t|\geq\delta} \mathrm{Real}(G_n(t)) = \min[\mathrm{Real}(G_n(\delta)), \mathrm{Real}(G_n(-\delta))],\ \forall\ n \geq 1,$$

where $G_n(t) = G_{n,\tau_n}(t)$.

(D). There exists $p, \ell > 0$ such that

$$\int_{-\infty}^{\infty} |\phi_n(\tau + it)/\phi_n(\tau)|^{\ell/n} = O(n^p)\quad \forall\ \tau \in I.$$

Then

$$k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi}\ \sigma} \qquad e^{-n\gamma_n(m_n)}[1 + O(|m_n - m|)]. \tag{3.3}$$

We have the following analogous theorem for lattice valued random variables.

Theorem 3.2. Let T_n take values in the set $\{a_n+kh_n: k=0,\pm1,\pm2, \ldots\}$. Let $\{m_n = (a_n+k_nh_n)/n\}$ be a sequence of real numbers, where $\{k_n\}$ is a sequence of integers. Assume that Conditions (A), (B) of Theorem

3.1 hold. Replace Conditions (C), (D) by the following:

(C′). There exists $\eta > 0$ such that for any $0 < \delta < \eta$,

$$\inf_{\delta \le |t| \le \pi/|h_n|} \mathrm{Real}(G_n(t)) = \min[\mathrm{Real}(G_n(\delta)),\ \mathrm{Real}(G_n(-\delta))],\ \forall\, n \ge 1.$$

(D′). There exists $p, \ell > 0$ such that

$$\int_{-\pi/h_n}^{\pi/h_n} |\phi_n(\tau + it)/\phi_n(\tau)| = O(n^p) \quad \forall\, \tau \in I.$$

Then

$$\frac{\sqrt{n}}{|h_n|} \Pr\left(\frac{T_n}{n} = m_n\right) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-n\gamma_n(m_n)}[1 + O(|m_n - m|)]. \tag{3.4}$$

We now proceed with some notation needed to state the main Theorem 3.4 of this section. Fix a sequence of random variables T_n, $n \ge 1$, satisfying the conditions of Theorem 3.1. Let ψ_n, γ_n be as defined in (3.1) and (3.2). For a probability measure P on $\mathbb{R}$, let $h(u)$ be the c.g.f. of P and $G_n(u) = \gamma_n(u) - h(u)$. The function G_n plays the same role as the function G of Section 2.

Definition 3.3. Let L_t be the class of all probability measures P on $\mathbb{R}$ such that the following two conditions hold:

$$\int e^{\psi_n(u)}\, dP(u) < \infty \quad \forall\, n \ge 1 \text{ and} \tag{3.5}$$

there exists $p, \ell > 0$ such that

$$\int e^{-\ell G_n(u)} du = O(n^p). \tag{3.6}$$

Let Y_r^* , $r \geq 1$ be a sequence of random variables with p.d.f. given by $d_r \exp[-c_{2r}y^{2r}/(h''(m))^{2r}(2r)!]$ if $r \geq 2$ and $N(0, h''(m)(h''(m) + c_2)/c_2)$ if $r = 1$, where c_{2r} is a constant and d_r is the normalizing factor. The following theorem is the main result of this section.

Theorem 3.4. Let $P \in L_t$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by

$$dQ_n(\underline{x}) = z_n^{-1} \phi_n(s_n/n) \prod_{j=1}^{n} dP(x_j). \tag{3.7}$$

Assume that G_n's have a unique global minimum of type r at the point m_n and $G_n^{(2r)}(m_n) \to c_{2r}$ as $n \to \infty$. Then

$$\frac{S_n - n\tau_n}{n^{1-\frac{1}{2r}}} \xrightarrow{d} Y_r^*. \tag{3.8}$$

Remark 3.5. The distribution function $Q_n(\underline{x})$ is well defined because

$$z_n = \int_{\mathbb{R}^n} \phi_n(s_n/n)\Pi dP(x_j)$$

$$= \int_{\mathbb{R}^n} e^{n\psi_n(s_n/n)} \Pi dP(x_j)$$

$$\leq \int_{\mathbb{R}^n} e^{\Sigma\psi_n(x_j)} \Pi dP(x_j) \quad \text{[since } \psi_n \text{ is convex]}$$

$$= \Big[\int_{\mathbb{R}} e^{\psi_n(x)} dP(x)\Big]^n < \infty \quad \text{[by (3.5)]}.$$

The proof of the above theorem is postponed until the end of Lemma 3.10. Let

$$g(y) = \exp[-y^{2r}c_{2r}/(2r)!] \tag{3.9}$$

and

$$g_n(y) = \sigma\sqrt{2\pi/n}\; k_n(m_n+n^{-\frac{1}{2r}}y)\; e^{n[h(m_n+n^{-\frac{1}{2r}}y)+G_n(m_n)]}, \tag{3.10}$$

where k_n is the p.d.f. of T_n/n, $n \geq 1$. We will need the following lemmas in the proof of Theorem 3.4. Lemma 3.6 shows that $g_n(y)$ converges to $g(y)$ as $n \to \infty$ for each y. Lemmas 3.7, 3.8, 3.9 and 3.10 show that

$$\int g_n(y) \to \int g(y) \text{ as } n \to \infty.$$

<u>Lemma 3.6.</u> Let $G_n^{(2r)}(m_n) = c_{2r,n}$. If G_n has a unique global minimum of type r at the point m_n and $c_{2r,n} \to c_{2r}$ as $n \to \infty$ then

$$g_n(y) \to g(y), \text{ as } n \to \infty. \tag{3.11}$$

<u>Proof</u>. Fix $y \in \mathbb{R}$. By Theorem 3.1 we have as $n \to \infty$,

$$\begin{aligned} g_n(y) &= \sigma\sqrt{2\pi/n}\; k_n(m_n+n^{-\frac{1}{2r}}y)e^{n[h(m_n+n^{-\frac{1}{2r}}y)+G_n(m_n)]} \\ &= e^{n[h(m_n+n^{-\frac{1}{2r}}y)+G_n(m_n)]-n\gamma_n(m_n+n^{-\frac{1}{2r}}y)}[1+O(|m_n+n^{-\frac{1}{2r}}y-m|)] \\ &= e^{-n[G_n(m_n+n^{-\frac{1}{2r}}y)-G_n(m_n)]}[1+O(|m_n+n^{-\frac{1}{2r}}y-m|)] \\ &= e^{-[y^{2r}c_{2r,n}/(2r)!+o(y^{2r}/n)]}[1+O(|m_n+n^{-\frac{1}{2r}}y-m|)]. \end{aligned}$$

The lemma is now immediate taking limits as $n \to \infty$. ||

The following Lemma 3.7 is crucial to the proof of the Lemma 3.8.

Lemma 3.7. Let $0 < \delta < \frac{1}{2r}$. Suppose that G_n's have unique global minimum of type r at the point m_n then there exists N such that

$$n[G_n(m_n+n^{-\frac{1}{2r}}y)-G_n(m_n)] \geq \frac{y^{2r}c_{2r}}{2(2r)!} \quad \forall\ n \geq N, \tag{3.12}$$

uniformly for $|y| < n^{\delta}$.

Proof. Let $0 < \varepsilon < c_{2r}/2$. Since $c_{2r,n}$ converges to c_{2r} we can find N_1 such that $c_{2r,n} > c_{2r}/2 + \varepsilon\ \forall\ n \geq N_1$. Also since

$$G_n(m_n+u)-G_n(m_n) = \frac{u^{2r}c_{2r,n}}{(2r)!} + o(|u|^{2r}) \text{ as } u \to 0$$

uniformly in n, we can find $\eta > 0$ such that for $|u| < \eta$ we have

$$G_n(m_n+u) - G_n(m_n) - \frac{u^{2r}c_{2r,n}}{(2r)!} > -\frac{u^{2r}\varepsilon}{(2r)!}.$$

Fix N_2 such that $n^{-1+2r\delta} < \eta\ \forall\ n \geq N_2$. Then $N = \max(N_1, N_2)$ does the job because for $n \geq N$ and $|y| < n^{\delta}$ we have $|y^{2r}/n| < \eta$. Therefore

$$\begin{aligned} n[G_n(m_n+n^{-\frac{1}{2r}}y)-G_n(m_n)] &= \left[\frac{y^{2r}c_{2r,n}}{(2r)!} + n\ o\left(\frac{|y|^{2r}}{n}\right)\right] \\ &\geq \left[\frac{y^{2r}}{(2r)!}\left(\frac{c_{2r}}{2}+\varepsilon\right) + \left(-\frac{y^{2r}\varepsilon}{(2r)!}\right)\right] \\ &= \frac{y^{2r}c_{2r}}{(2r)!2}, \end{aligned}$$

$\forall\, n \geq N$ uniformly for $|y| < n^{\delta}$. This completes the proof of the lemma. ||

<u>Lemma 3.8</u>. Let $0 < \delta < \frac{1}{2}r$. Let g and g_n be as defined by (3.9) and (3.10). Suppose that G_n has a unique global minimum of type r at the point m_n then

$$\int_{|y| \leq n^{\delta}} g_n(y) \to \int g(y)dy \text{ as } n \to \infty. \tag{3.13}$$

<u>Proof</u>. For $|y| \leq n^{\delta}$, note that $n^{-\frac{1}{2}r}y$ converges to zero uniformly in y. Therefore from Theorem 3.1 we get as $n \to \infty$

$$K_n(m_n + n^{-\frac{1}{2}r}y) = \sqrt{n/2\pi}\,\sigma^{-1} e^{-n\gamma_n(m_n + n^{-\frac{1}{2}r}y)}[1 + O(|m_n + n^{-\frac{1}{2}r}y - m|)]. \tag{3.14}$$

Thus

$$\int_{|y| \leq n^{\delta}} g_n(y)dy = \sigma\sqrt{2\pi/n} \int_{|y| \leq n^{\delta}} e^{n[h(m_n + n^{-\frac{1}{2}r}y) + G_n(m_n)]} k_n(m_n + n^{-\frac{1}{2}r}y)dy \tag{3.15}$$

$$= \int_{|y| \leq n^{\delta}} e^{-n[G_n(m_n + n^{-\frac{1}{2}r}y) - G_n(m_n)]}[1 + O(|m_n + n^{-\frac{1}{2}r}y - m|)]dy$$

$$= \int \lambda_n(y)dy,$$

where

$$\lambda_n(y) = I(|y| \leq n^{\delta})e^{-n[G_n(m_n + n^{-\frac{1}{2}r}y) - G_n(m_n)]}[1 + O(|m_n + n^{-\frac{1}{2}r}y - m|)]$$

and I is the indicator function. It follows from Lemma 3.7 that $|\lambda_n(y)|$ is bounded by an integrable function. We can now conclude from Lemma 3.6 and Lebesgue dominated convergence theorem that

$$\int \lambda_n(y)dy \to \int g(y)dy \text{ as } n \to \infty. \tag{3.16}$$

The proof is now complete. ||

The following Lemma 3.9 is needed to prove the next Lemma 3.10.

Lemma 3.9. Let $\{T_n, n \geq 1\}$ be a sequence of random variables satisfying the conditions of Theorem 3.1. Then

$$e^{n\gamma_n(m_n+y)}k_n(m_n+y) = O(n^{P+1}) \quad \forall\, y, \text{ as } n \to \infty. \tag{3.17}$$

Proof. Let $y \in \mathbb{R}$ and τ_y be such that

$$\gamma_n(m_n+y) = (m_n+y)\tau_y - \psi_n(\tau_y).$$

A simple application of the inversion formula applied to k_n, yields

$$\left|e^{n\gamma_n(m_n+y)}k_n(m_n+y)\right| = \left|\frac{n}{2\pi}\int_{-\infty}^{\infty} e^{n[\psi_n(\tau_y+it)-\psi_n(\tau_y)-itn(m_n+y)]}dt\right|$$

$$\leq \frac{n}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\phi_n(\tau_y+it)}{\phi_n(\tau_y)}\right|^{1/n} dt = O(n^{P+1}). \;||$$

Lemma 3.10. Let $0 < \delta < \frac{1}{2}r$. Suppose that G_n has a unique global minimum at the point m_n and let g_n be as defined by (3.10) then

$$\int_{|y|>n^{\delta}} g_n(y)dy \to 0 \quad \text{as } n \to \infty. \tag{3.18}$$

Proof. By (3.10) we have

$$\int_{|y|>n^{\delta}} g_n(y)\,dy$$

$$= \sigma\sqrt{2\pi/n} \int_{|y|>n^{\delta}} e^{n[h(m_n+n^{-\frac{1}{2}r}y)+G_n(m_n)]} k_n(m_n+n^{-\frac{1}{2}r}y)\,dy$$

$$= \sigma\sqrt{2\pi/n} \int_{|y|>n^{\delta}} e^{-n[G_n(m_n+n^{-\frac{1}{2}r}y)-G_n(m_n)]+n\gamma_n(m_n+n^{-\frac{1}{2}r}y)} k_n(m_n+n^{-\frac{1}{2}r}y)\,dy.$$

Substituting $y' = n^{-\frac{1}{2}r}y$, we get

$$\left|\int_{|y|>n^{\delta}} g_n(y)\,dy\right|$$

$$\le \sigma\sqrt{2\pi}\; n^{-\frac{1}{2}(1-\frac{1}{r})} \int_{|y|\ge n^{\delta-\frac{1}{2}r}} |e^{-n[G_n(m_n+y)-G_n(m_n)]}|\,|e^{n\gamma_n(m_n+y)} k_n(m_n+y)|\,dy$$

$$\le O(n^{P+\frac{1}{2}(1+\frac{1}{r})}) \max_{|y|\ge n^{\delta-\frac{1}{2}r}} e^{-(n-\ell)[G_n(m_n+y)-G_n(m_n)]} \int e^{-\ell[G_n(m_n+y)-G_n(m_n)]}\,dy.$$

The last inequality follows from Lemma 3.9. Thus we get from Condition (3.6)

$$\left|\int_{|y|>n^{\delta}} g_n(y)\,dy\right| \le O(n^{q}) \max_{|y|\ge n^{\delta-\frac{1}{2}r}} e^{-(n-\ell)[G_n(m_n+y)-G_n(m_n)]}$$

$$= O(n^{q})e^{-(n-\ell)L_n}, \text{ where } q = 2P+\tfrac{1}{2}(1+\tfrac{1}{r})$$

and

$$L_n = \min_{|y|\geq n^{\delta-\frac{1}{2}r}} [G_n(m_n+y)-G_n(m_n)]$$

$$= \min\{[G_n(n^{\delta-\frac{1}{2}r}+m_n)-G_n(m_n)],\ [G_n(-n^{\delta-\frac{1}{2}r}+m_n)-G_n(m_n)]\}$$

$$= \frac{c_{2r,n}}{(2r)!} n^{2r(\delta-\frac{1}{2}r)} + o(n^{2r(\delta-\frac{1}{2}r)}),$$

since m_n is the unique global minimum of G_n and $c_{2r,n} > 0$. Hence

$$\left| \int_{|y|>n^\delta} g_n(y)dy \right| \leq O(n^q)e^{-(n-\ell)\left[\frac{c_{2r,n}}{(2r)!} n^{2r(\delta-\frac{1}{2}r)}+o(n^{2r(\delta-\frac{1}{2}r)})\right]}$$

which goes to 0 since $0 < \delta < \frac{1}{2}r$ and the proof is complete. ||

Going back to the proof of Theorem 3.4 we first express dQ_n defined in (3.7) as follows:

$$dQ_n(\underline{x}) = z_n^{-1}\phi_n(s_n/n)\Pi dP(x_j)$$

$$= z_n^{-1} \int e^{ys_n} k_n(y)dy \Pi dP(x_j)$$

$$= z_n^{-1}n^{-\frac{1}{2}r} \int e^{(m_n+n^{-\frac{1}{2}r}y)s_n} k_n(m_n+n^{-\frac{1}{2}r}y)dy\Pi dP(x_j)$$

$$= z_n^{-1}n^{-\frac{1}{2}r} \int \prod_{j=1}^{n} e^{x_j(m_n+n^{-\frac{1}{2}r}y)-h(m_n+n^{-\frac{1}{2}r}y)} dP(x_j)$$

$$\times e^{nh(m_n+n^{-\frac{1}{2}r}y)} k_n(m_n+n^{-\frac{1}{2}r}y)dy$$

$$= \int \prod_{j=1}^{n} dM_{n,y}(x_j)f_n(y)dy, \qquad (3.19)$$

where

$$dM_{n,y}(x_j) = e^{x_j(m_n+n^{-\frac{1}{2}r}y)-h(m_n+n^{-\frac{1}{2}r}y)}\, dP(x_j) \qquad (3.20)$$

and

$$f_n(y) = z_n^{-1} n^{-\frac{1}{2}r} e^{nh(m_n+n^{-\frac{1}{2}r}y)} k_n(m_n+n^{-\frac{1}{2}r}y). \qquad (3.21)$$

Since $\int dQ_n(\underline{x}) = 1$ and $\int dM_{n,y}(x_j) = 1$ for each y and j we have $\int f_n(y)dy = 1$. Thus we can introduce random variables V_n with p.d.f. f_n and the representation (3.19) of $dQ_n(\underline{x})$ shows that $X_j^{(n)}$, $j = 1, \ldots, n$, are i.i.d. $dM_{n,y}(x)$ given $V_n = y$. We now proceed and obtain the limiting distribution of $(S_n-n\tau_n)/n^{1-\frac{1}{2}r}$ under $dM_{n,y}(x)$. Consider,

$$\log E_{M_{n,y}} e^{t[(S_n-n\tau_n)/n^{1-\frac{1}{2}r}]}$$

$$= n\left[-\frac{t\tau_n}{n^{1-\frac{1}{2}r}} + h\left(\frac{t}{n^{1-\frac{1}{2}r}} + n^{-\frac{1}{2}r}y + m_n\right) - h(n^{-\frac{1}{2}r}y+m_n)\right]$$

$$= n\left[-\frac{t\tau_n}{n^{1-\frac{1}{2}r}} + h'(n^{-\frac{1}{2}r}y+m_n)\ \frac{t}{n^{1-\frac{1}{2}r}} + \frac{t^2}{2n^{2-1/r}}\, h''(n^{-\frac{1}{2}r}y+m_n)+o(n^{-1})\right]$$

$$= n\left[-\frac{t\tau_n}{n^{1-\frac{1}{2}r}} \pm \frac{h'(m_n)t}{n^{1-\frac{1}{2}r}} + \frac{h''(m_n)ty}{n} + \frac{t^2}{2n^{2-1/r}}\, h''(m_n)+o(n^{-1})\right]$$

$$=\left[h''(m_n)ty + \frac{h''(m_n)t^2}{2n^{1-1/r}} +o(1)\right],$$

since $G_n'(m_n) = 0$ and therefore $\gamma_n'(m_n) = h'(m_n) = \tau_n$. Thus

$$\log E_{M_{n,y}} e^{t\left[\frac{S_n - n\tau_n}{n^{1-\frac{1}{2r}}}\right]} \to \begin{cases} h''(m)ty & \text{if } r > 1 \\ h''(m)ty + \frac{h''(m)t^2}{2} & \text{if } r = 1 \end{cases} . \tag{3.22}$$

This shows that the limiting distribution of $(S_n - n\tau_n)/n^{1-\frac{1}{2r}}$ given $V_n = y$ is degenerate at $h''(m)y$ if $r > 1$ and $N(h''(m)y,\ h''(m))$ if $r = 1$.

Next we note that

$$f_n(y) = z_n^{-1} n^{-\frac{1}{2r}} e^{nh(m_n + n^{-\frac{1}{2r}}y)} K_n(m_n + n^{-\frac{1}{2r}}y)$$

$$= \frac{g_n(y)}{\int g_n(y)\,dy},$$

where $g_n(y)$ is as defined in (3.10). By Lemmas (3.6), (3.8) and (3.10) it follows that

$$f_n(y) \to f(y) = \frac{g(y)}{\int g(y)\,dy} \quad \text{as } n \to \infty, \tag{3.23}$$

where

$$g(y) = e^{-y^{2r} c_{2r}/(2r)!}.$$

The proof of the theorem is completed applying Theorem 3.12 of Sethuraman to (3.22) and (3.23). ||

Remark 3.11. When T_n is the sum of n i.i.d. random variables distributed as U with m.g.f. ϕ, $\phi_n(s_n/n)$ becomes $\phi^n(s_n/n)$ and the class of probability measures L_t reduces to the class L_u. Thus Theorem 3.4 generalizes Theorem 2.11 to a larger class of Hamiltonians and probability measures.

We now state the theorem of Sethuraman (1961) which was crucially used to obtain the limiting marginal distribution of $(S_n-n\tau_n)/n^{1-\frac{1}{2}r}$ in the proof of Theorem 3.4.

Theorem 3.12 (Sethuraman). Let λ_n be a sequence of probability measures on $V \times W$, where V and W are topological spaces. Let μ_n be the marginal probability measure of λ_n on V and $\nu_n(v, \cdot)$ be the conditional probability measure on W. Suppose that μ_n converges to a probability measure μ for every measurable set in V and for almost all v with respect to μ, $\nu_n(v, \cdot)$ converges weakly to $\nu(v, \cdot)$. Then λ_n converges weakly to λ, where

$$\lambda(A \times B) = \int_A \nu(v, B)d\mu(v) \qquad (3.24)$$

for every measurable rectangular set $A \times B$.

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A triangular array of dependent random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ whose joint distribution is given by $dQ_n(\underline{x}) = z_n^{-1} \exp[-H_n(\underline{x})]\Pi dP(x_j)$, where $\underline{x}(x_1,\ldots,x_n) \in \mathbb{R}^n$, z_n is the normalizing constant and P is a probability measure on $\mathbb{R}$ has been used to describe the distribution of magnetic spins in a body. Let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$ be the total magnetism present in the body. For certain forms of the function H, Ellis and Newman (Z. Wahrscheinlichkeistheorie and Verw. Gebiete 44 (1978) 117-139)

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ı and Sethuraman (IMS Bulletin (1978) Abstract # 165-116)

: appropriate conditions on P, there exists an integer

$_n/n^{1-\frac{1}{2r}}$ converges in distribution to a random variable

ı for $r = 1$ and non-Gaussian for $r \geq 2$. In this paper

'ge deviation local limit theorems for arbitrary sequences

.es of Chaganty and Sethuraman (Dept. of Stat., FSU,

) we obtain similar central limit theorems for a wider

s H_n, thus generalizing the results of the previous


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